

The Laplacian lattice of a graph under a simplicial distance function

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Abstract

We provide a complete description of important geometric invariants of the Laplacian lattice of a multigraph under the distance function induced by a regular simplex, namely Voronoi Diagram, Delaunay Triangulation, Delaunay Polytope and its combinatorial structure, Shortest Vectors, Covering and Packing Radius. We use this information to obtain the following results: i. Every multigraph defines a Delaunay triangulation of its Laplacian lattice and this Delaunay triangulation contains complete information of the multigraph up to isomorphism. ii. The number of multigraphs with a given Laplacian lattice is controlled, in particular upper bounded, by the number of different Delaunay triangulations. iii. We obtain formulas for the covering and packing densities of a Laplacian lattice and deduce that in the space of Laplacian lattices of undirected connected multigraphs, the Laplacian lattices of highly connected multigraphs such as Ramanujan multigraphs possess good covering and packing properties.

1 Introduction

The work presented in this paper came as an attempt to obtain a thorough understanding of the Laplacian lattice of a multigraph, the lattice generated by the rows of the Laplacian matrix of the multigraph under a simplicial distance function. This study played a central role in the geometric proof of the Riemann-Roch theorem for graphs obtained in [2] where the study of the Laplacian lattice under a simplicial distance function was initiated. The study was focussed towards obtaining a geometric proof of the Riemann-Roch theorem for graphs. In this paper, we provide a complete description of important geometric invariants of the Laplacian lattice of a multigraph under the distance function induced by a regular simplex. In particular, we obtain a combinatorial interpretation for important geometric invariants of the Laplacian lattice under the simplicial distance function. We use this information to obtain the following results for the Laplacian lattice under the simplicial distance function:

1. Every connected multigraph defines a Delaunay triangulation of its Laplacian lattice and this Delaunay triangulation contains complete information of the multigraph up to isomorphism.
2. The number of multigraphs with a given Laplacian lattice is controlled, in particular upper bounded, by the number of different Delaunay triangulations.
3. We obtain formulas for the covering and packing densities of a Laplacian lattice. Furthermore, we use the combinatorial interpretation of the geometric invariants of the

Laplacian lattice to relate the connectivity properties of the multigraph to the covering and packing density of the corresponding Laplacian lattice in the simplicial distance function. In particular, we show that in the space of Laplacian lattices of undirected connected multigraphs, Laplacian lattices of multigraphs with high-connectivity such as Ramanujan graphs possess good packing and covering density.

Let us provide some motivation for studying firstly the Laplacian lattice and secondly, for studying its properties under the simplicial distance function¹. The answer is rooted back to the pioneering work of Baker and Norine [4] where analogues of the Riemann-Roch and Abel-Jacobi theory on algebraic curves were developed on graphs and the geometric of proof of the Riemann-Roch theorem obtained in [2]. A solitary game played on an undirected connected multigraph known as the chip firing game played a key role in these works and is defined as follows: Each vertex of the graph is assigned an integer (positive or negative), that can be thought of as “chips” and this assignment is called the initial configuration. At each move of the game, an arbitrary vertex v is allowed to either lend or borrow one chip along each edge incident with it and we obtain a new configuration. We define two configurations C_1 and C_2 to be equivalent if C_1 can be reached from C_2 by a sequence of chip firings. The Laplacian matrix $Q(G)$ of the graph naturally comes into the picture as follows:

Lemma 1. *Configurations C_1 and C_2 are equivalent if and only if $C_1 - C_2$ can be expressed as $Q(G) \cdot w$ for some vector w with integer coordinates.*

Some natural questions on such a game are:

1. Is a given configuration equivalent to an effective configuration i.e. a configuration where each vertex has a non-negative number of chips?
2. More generally, given a configuration what is the minimum number of chips that must be removed from the system so that the resulting configuration is not equivalent to an effective configuration?

Let us translate the above notions to the geometric language of lattices and distance functions. The set of all configurations that are equivalent to the origin is the set $\{Q(G) \cdot x \mid x \in \mathbb{Z}^{n+1}\}$ where $n + 1$ is the number of vertices of the multigraph and this set is precisely the Laplacian lattice L_G of the multigraph. Let $H_q^+ = \{y \in \mathbb{Z}^{n+1} \mid y \geq q\}$ where, for points in \mathbb{Z}^{n+1} we say that $x \geq y$ if $x_i \geq y_i$ for all i from 1 to $n + 1$ and say that x dominates y . The set of effective configurations is precisely H_O^+ where O is the origin. Fix a positive real number k , consider the set E_k of configurations that are equivalent to an effective configuration and with total number of chips being equal to k . In geometric terms, the set E_k is precisely the set of points with integer coordinates contained in the intersection of the hyperplane H_k given by the equation $\sum_{i=1}^{n+1} x_i = k$ and the union of cones H_q^+ over all points q in the Laplacian lattice. A simple calculation shows that the intersection of H_O^+ with the hyperplane H_k is a regular simplex with centroid at the origin. As a consequence, the set E_k is precisely the set of points in H_k with integer coordinates and whose projection onto H_0 (the hyperplane containing L_G) is contained in an arrangement of regular simplices with centroid at points in the Laplacian lattice. Hence, the definition of effectiveness has an underlying distance function, namely the simplicial distance function. This observation motivated us to study the Laplacian lattice under the simplicial distance function.

¹Let S be a full dimensional simplex in \mathbb{R}^n that contains the origin, that we call the center of S , in its interior. For two points $P_1, P_2 \in \mathbb{R}^n$, the simplicial distance function induced by S is the smallest factor by which we need to scale the translated copy of S centered at P_1 so that it contains P_2 . The Euclidean distance can be obtained by replacing the simplex by a sphere centered at the origin. Given a lattice, the simplicial distance function of a point with respect to a lattice is the minimum simplicial distance of the point to any lattice point.

1.1 Related Work

A substantial body of work is devoted to the study of lattices constructed from graphs, the most well studied ones being the lattice of integral cuts and the lattice of integral flows. This line of investigation was pioneered by the work of Bacher, Harpe and Nagnibeda [3] where they provide a combinatorial interpretation of various parameters of the lattice of integral flows and the lattice of integral cuts in the Euclidean distance function, for example they show that the square norm of the shortest vector of the lattice of integral flows is equal to the girth of the graph. In contrast, the study of the Laplacian lattice under simplicial distance function gives rise to some new and interesting phenomenon, that are not seen in the Euclidean case and in some cases provides more refined information about the graph. Here we note such instances:

i. The Laplacian lattice under the simplicial distance function has different Delaunay triangulations and they contain combinatorial information i.e. the number of different Delaunay triangulations upper bounds the number of multigraphs with the given lattice as their Laplacian lattice.

ii. The Delaunay triangulation under the simplicial distance function provides more refined information on the underlying (connected) multigraph and in fact characterizes the multigraph completely up to isomorphism. On the other hand in the Euclidean case such an analogue is true only for three-connected graphs. A weaker result is known for general connected graphs [29].

iii. The absence of edges between vertices shows up as “degeneracies” in the Voronoi diagram of the Laplacian lattice with respect to the simplicial distance function. In particular, multigraphs where every pair of vertices are connected by an edge turn out to be the “nicest” cases for the Voronoi diagram of the Laplacian lattice. See Section 3.2 for more details.

On the computational side, Sikric, Schürmann and Vallentin [26] provide an algorithm to compute the Voronoi cell of a lattice. Also, see the paper [10] for a connection between Voronoi diagrams of lattices and certain regular arrangement of hyperplanes called lattice dicings. In each section, we will further provide references to works that are most relevant to the results in that section.

1.2 Organisation

Let us briefly discuss the organisation of the rest of the paper. Section 2 describes some preliminaries and builds up notation useful to develop our main results. We describe the Voronoi neighbours (Section 3), the packing and covering radius (Section 4) and the length of the shortest vector of the Laplacian lattice under the simplicial distance function (Section 5). We then study applications of these results: in Section 6 we show that the Delaunay polytope of the Laplacian lattice under the simplicial distance function characterizes the graph completely up to isomorphism, in Section 7 we treat the problem of counting the number of graphs with a given Laplacian lattice and in Section 8 we study packing and covering problems on Laplacian lattices.

2 Preliminaries

In this section, we collect some basic definitions and results that will be useful in the rest of the paper.

2.1 Lattices

A **lattice** L is a discrete subgroup of the Euclidean vector space \mathbb{R}^n . More concretely, a lattice is the Abelian group obtained by taking all the integral combinations of a set of linearly independent vectors b_1, \dots, b_k in \mathbb{R}^n . More precisely,

$$L = \left\{ \sum_{i=1}^k \alpha_i b_i \mid \alpha_i \in \mathbb{Z} \right\} \quad (1)$$

The set $\mathcal{B} = \{b_1, \dots, b_k\}$ is called a basis of L and the integer k is called the dimension of L , denoted by $\dim(L)$, is independent of the choice of the basis. We denote the subspace spanned by the elements of \mathcal{B} by $\text{Span}(L)$.

Let us now look at the important geometric invariants of a lattice. The volume of the lattice $\text{Vol}(L)$, also known as the discriminant or determinant, is defined as $\sqrt{\det(\mathcal{B}\mathcal{B}^t)}$ where the basis \mathcal{B} is represented as a matrix with its elements row-wise and hence, $\mathcal{B}\mathcal{B}^t$ is the Gram matrix of the basis elements. Another important invariant is the norm of the shortest vector of a lattice.

Definition 1. *The shortest vector of L in the Euclidean norm is an element $q \neq (0, \dots, 0)$ of L such that $q \cdot q \leq q' \cdot q'$ for all $q' \in L$ and we denote $\|q\|_2$ as $\nu_E(L)$.*

Two other important invariants of a lattice are the packing radius and the covering radius.

Definition 2. *Let $B(q, R)$ be the Euclidean ball centered at the point $q \in \mathbb{R}^n$ and with radius R . The packing radius $\text{Pac}_E(L)$ and the covering radius $\text{Cov}_E(L)$ of a lattice L is defined as:*

$$\begin{aligned} \text{Pac}_E(L) &= \sup\{R \mid B(q_1, R) \cap B(q_2, R) = \emptyset \ \forall q_1, q_2 \in L, q_1 \neq q_2\}, \\ \text{Cov}_E(L) &= \inf\{R \mid \forall p \in \text{Span}(L) \ \exists q \in L : p \in B(q, R)\}. \end{aligned}$$

The shortest vector and the packing radius are related by $\text{Pac}_E(L) = \nu_E(L)/2$. The study of relations between geometric invariants of a lattice is a classical topic [27]. To convey the flavour of the topic, we state a basic result known as Minkowski's First Theorem:

Theorem 1. *(Minkowski [19]) A lattice L has a non-zero element whose Euclidean norm is upper bounded by $2\text{Vol}^{1/n}(L)/V_E^{1/n}$ where V_E is the volume of the unit ball in $\text{Span}(L)$.*

2.2 The Laplacian Lattice of a Graph

For an undirected connected multigraph G , the Laplacian matrix $Q(G)$ is defined as $D(G) - A(G)$ where $D(G)$ is the diagonal matrix with the degree of every vertex in its diagonal and $A(G)$ is the vertex-adjacency matrix of the graph. We assume the following standard form of the Laplacian matrix:

$$Q = \begin{bmatrix} \delta_0 & -b_{01} & -b_{02} \dots & -b_{0n} \\ -b_{10} & \delta_1 & -b_{12} \dots & -b_{1n} \\ \vdots & \vdots & \ddots & \\ -b_{n0} & -b_{n1} & -b_{n2} \dots & \delta_n \end{bmatrix} \quad (2)$$

has the following properties:

(C_1) b_{ij} 's are integers, $b_{ij} \geq 0$ for all $0 \leq i \neq j \leq n$ and $b_{ij} = b_{ji}$, $\forall i \neq j$.

(C₂) $\delta_i = \sum_{j=1, j \neq i}^n b_{ij} = \sum_{j=1, j \neq i}^n b_{ji}$ (and is the degree of the i -th vertex).

Remark 1. *Note that we consider undirected connected multigraphs without self-loops and hence, there can be multiple edges between vertices. In the rest of paper, when we refer to graphs we actually mean multigraphs.*

Though the Laplacian matrix of a graph contains essentially the same information as the adjacency matrix of a graph, it enjoys other nice properties. See, for example, the chapter “The Laplacian matrix of a Graph” in the algebraic graph theory book of Godsil and Royle [14] for a more complete discussion.

Lemma 2. *The Laplacian matrix $Q(G)$ is a symmetric positive semi-definite matrix.*

Another remarkable property of the Laplacian matrix is described in the Matrix-Tree theorem:

Theorem 2. (Kirchoff’s Matrix-Tree Theorem) *The absolute of value of any cofactor is equal to the number of spanning trees of the graph.*

A remarkable aspect of the matrix-tree theorem is that it reduces counting the number of spanning trees of a graph into a determinant computation and hence provides a polynomial time algorithm for it.

Definition 3. (The Laplacian lattice of a graph) *Given the Laplacian matrix $Q(G)$, the lattice generated by the rows (or equivalently the columns) of $Q(G)$ is called the Laplacian lattice L_G of the graph.*

Definition 4. (The Root Lattice A_n) *The root lattice ² A_n is the lattice of integer points in the hyperplane $H_0 = \{(x_0, \dots, x_n) \mid \sum_{i=0}^{n+1} x_i = 0, x_i \in \mathbb{R}\}$. More precisely,*

$$A_n = \{(x_0, \dots, x_n) \mid \sum_{i=0}^{n+1} x_i = 0, x_i \in \mathbb{Z}\}.$$

We now make a few simple observations on the Laplacian lattice of a graph.

Lemma 3. *The Laplacian lattice of a graph on $n + 1$ -vertices is a sublattice of the root lattice A_n .*

Definition 5. (The covolume of a sublattice) *A full-dimensional sublattice L_s of a lattice L is a subgroup of the Abelian group L the cardinality of the quotient group L/L_s is called the covolume of L_s with respect to L .*

Lemma 4. [25] *The covolume of the Laplacian lattice of G with respect to A_n is equal to the number of spanning trees of G .*

The elements of A_n/L_G naturally possess an Abelian group structure and this group is known as the Picard group of G , also known as the Jacobian of G . A number of works have been devoted to the study of the structure of this group and the information that it contains about the underlying graph, see for example the works of Biggs [6], Kotani and Sunada [17] and Lorenzini [18]. As a straightforward corollary to Lemma 4 we obtain:

Corollary 1. *The cardinality of the Picard group of G is equal to the number of spanning trees of G .*

²Root refers here to root systems in the classification theory of simple Lie algebras [7]

2.3 Polyhedral Distance Functions

Let \mathcal{P} be a convex polytope in \mathbb{R}^n with the reference point $O = (0, \dots, 0)$ that we call the “center” in its interior. By $\mathcal{P}(p, \lambda)$ we denote a dilation of \mathcal{P} by a factor λ and its center translated to the point p i.e. $\mathcal{P}(p, \lambda) = p + \lambda \mathcal{P}$ and $\lambda \mathcal{P} = \{ \lambda x \mid x \in \mathcal{P} \}$. We define the \mathcal{P} -midpoint of two points p and q in \mathbb{R}^n as $\inf\{R \mid \mathcal{P}(p, R) \cap \mathcal{P}(q, R) \neq \emptyset\}$. The *polyhedral distance function* $d_{\mathcal{P}}(\cdot, \cdot)$ between the points of \mathbb{R}^n is defined as follows:

$$\forall p, q \in \mathbb{R}^n, d_{\mathcal{P}}(p, q) := \inf\{\lambda \geq 0 \mid q \in \mathcal{P}(p, \lambda)\}.$$

$d_{\mathcal{P}}$ is not generally symmetric, indeed it is easy to check that $d_{\mathcal{P}}(\cdot, \cdot)$ is symmetric if and only if the polyhedron \mathcal{P} is centrally symmetric i.e. $\mathcal{P} = -\mathcal{P}$. Nevertheless $d_{\mathcal{P}}(\cdot, \cdot)$ satisfies the triangle inequality.

Lemma 5. *For every three points $p, q, r \in \mathbb{R}^n$, we have $d_{\mathcal{P}}(p, q) + d_{\mathcal{P}}(q, r) \geq d_{\mathcal{P}}(p, r)$. In addition, if q is a convex combination of p and r , then $d_{\mathcal{P}}(p, q) + d_{\mathcal{P}}(q, r) = d_{\mathcal{P}}(p, r)$.*

We also observe that the polyhedral metric $d_{\mathcal{P}}(\cdot, \cdot)$ is translation invariant, i.e.

Lemma 6. *For any two points p, q in \mathbb{R}^n , and for any vector $v \in \mathbb{R}^n$, we have $d_{\mathcal{P}}(p, q) = d_{\mathcal{P}}(p - v, q - v)$. In particular, $d_{\mathcal{P}}(p, q) = d_{\mathcal{P}}(p - q, O) = d_{\mathcal{P}}(O, q - p)$.*

Remark 2. The notion of a polyhedral distance function is essentially the concept of a gauge function of a convex body that has been studied in [27]. Lemmas 5 and 6 can be derived in a straightforward way from the results in [27].

Recall that the n -dimensional hyperplane H_0 is defined as $H_0 = \{(x_0, \dots, x_n) \mid \sum_{i=0}^n x_i = 0\}$. We will be mainly be using distance functions defined by regular simplices Δ and $\bar{\Delta}$ where the simplices Δ and $\bar{\Delta}$ are defined as follows:

Definition 6. *The regular simplex Δ is the convex hull of $t_0, \dots, t_n \in H_0$ where*

$$t_{ij} = \begin{cases} n, & \text{if } i = j, \\ -1, & \text{otherwise} \end{cases}$$

for i from $1, \dots, n$ and t_{ij} is the j -th coordinate of t_i . We define $\bar{\Delta}$ as $-\Delta$.

We note that for points in H_0 , the distance functions d_{Δ} and $d_{\bar{\Delta}}$ have a simple formula:

Lemma 7. *(Lemma 4.7, [2]) For any pair of points p, q in H_0 , we have:*

$$\begin{aligned} d_{\Delta}(p, q) &= |\min_i (q_i - p_i)|, \\ d_{\bar{\Delta}}(p, q) &= |\min_i (p_i - q_i)|. \end{aligned}$$

Remark 3. *The notion of a simplicial distance function with respect to a lattice is sometimes captured in the language of “lattice point free simplices”, see [22] for more details on this viewpoint. The author is indebted to Bernd Sturmfels for pointing out this connection.*

2.4 Geometric Invariants of a Lattice with respect to Polyhedral Distance Functions

In Section 2.1, we defined some important geometric invariants of a lattice with respect to the Euclidean norm. We define analogous invariants with respect to polyhedral distance functions.

Definition 7. An element q of L is called a shortest vector with respect to the polyhedral distance function $d_{\mathcal{P}}$ if $d_{\mathcal{P}}(O, q) \leq d_{\mathcal{P}}(O, q')$ for all $q' \in L \setminus \{O\}$, where O is the origin. We denote $d_{\mathcal{P}}(O, q)$ by $\nu_{\mathcal{P}}(L)$.

Note that the shortest vector with respect the distance functions $d_{\mathcal{P}}$ and $d_{\bar{\mathcal{P}}}$ can be potentially different.

Definition 8. For a lattice L , we define packing and covering radius of L with respect \mathcal{P} as:

$$\text{Pac}_{\mathcal{P}}(L) = \sup\{R \mid \mathcal{P}(q_1, R) \cap \mathcal{P}(q_2, R) = \emptyset, \forall q_1, q_2 \in L, q_1 \neq q_2\}$$

$$\text{Cov}_{\mathcal{P}}(L) = \inf\{R \mid \forall p \in \text{Span}(L) \text{ is contained in } \mathcal{P}(q, R) \text{ for some } q \in L\}$$

As we shall observe, for polyhedral distance functions it is no longer true that $\text{Pac}_{\mathcal{P}}(L) = \nu_{\mathcal{P}}(L)/2$.

2.4.1 Voronoi Diagrams

Consider a discrete subset \mathcal{S} in \mathbb{R}^n . For a point s in \mathcal{S} , we define the *Voronoi cell* of s with respect to the distance function $d_{\mathcal{P}}$ as $V_{\mathcal{P}}(s) = \{p \in \mathbb{R}^n \mid d_{\mathcal{P}}(p, s) \leq d_{\mathcal{P}}(p, s') \text{ for any other point } s' \in \mathcal{S}\}$. Unlike the Euclidean case, the Voronoi cell of a point is not necessarily a convex set, see Figure 1 for an example, but nevertheless the following weaker property holds.

Lemma 8. Let \mathcal{S} be a discrete subset of \mathbb{R}^n and $\text{Vor}_{\mathcal{P}}(\mathcal{S})$ be the Voronoi cell decomposition of \mathbb{R}^n . For any point s in \mathcal{S} , the Voronoi cell $V_{\mathcal{P}}(s)$ is a star-shaped polyhedron with s as a kernel.

Lemma 9. Let \mathcal{S} be a discrete subset of \mathbb{R}^n . For any point s in \mathcal{S} we have: $\text{Vor}_{\mathcal{P}}(s) = -\text{Vor}_{-\mathcal{P}}(s)$.

Definition 9. (Voronoi Neighbours) We say that distinct points $p, q \in \mathcal{S}$ are Voronoi neighbours if the intersection of their Voronoi cells are non-empty.

Lemma 10. (Lemma 4.6 of [2]) For the Voronoi cell decomposition of a full-dimensional lattice in \mathbb{R}^n with respect to a polyhedral distance function $d_{\mathcal{Q}}$, the Voronoi cell of every lattice point is compact.

Lemma 11. Any shortest vector of a lattice L under the polyhedral distance function $d_{\mathcal{P}}$ is a Voronoi neighbour of the origin under the same distance function.

Proof. Let p be a shortest vector of L in the distance function $d_{\mathcal{P}}$. Consider the intersection p_I of the ray \overrightarrow{Op} with the Voronoi cell $V_{\mathcal{P}}(O)$ of the origin under the distance function $d_{\mathcal{P}}$ and by Lemma 10, we know that the ray \overrightarrow{Op} intersects $V_{\mathcal{P}}(O)$ at some point $p_I \neq p$. By Lemma 5, we have: $d_{\mathcal{P}}(O, p) = d_{\mathcal{P}}(O, p_I) + d_{\mathcal{P}}(p_I, p)$. Assume for contradiction that p is not a Voronoi neighbour of the origin under the distance function $d_{\mathcal{P}}$. This means that there is a lattice point p' such that $d_{\mathcal{P}}(p_I, p') < d_{\mathcal{P}}(p_I, p)$. Hence we have:

$$d_{\mathcal{P}}(O, p') \leq d_{\mathcal{P}}(O, p_I) + d_{\mathcal{P}}(p_I, p') < d_{\mathcal{P}}(O, p_I) + d_{\mathcal{P}}(p_I, p) = d_{\mathcal{P}}(O, p). \quad (3)$$

This contradicts our assumption that p is a shortest vector of L . \square

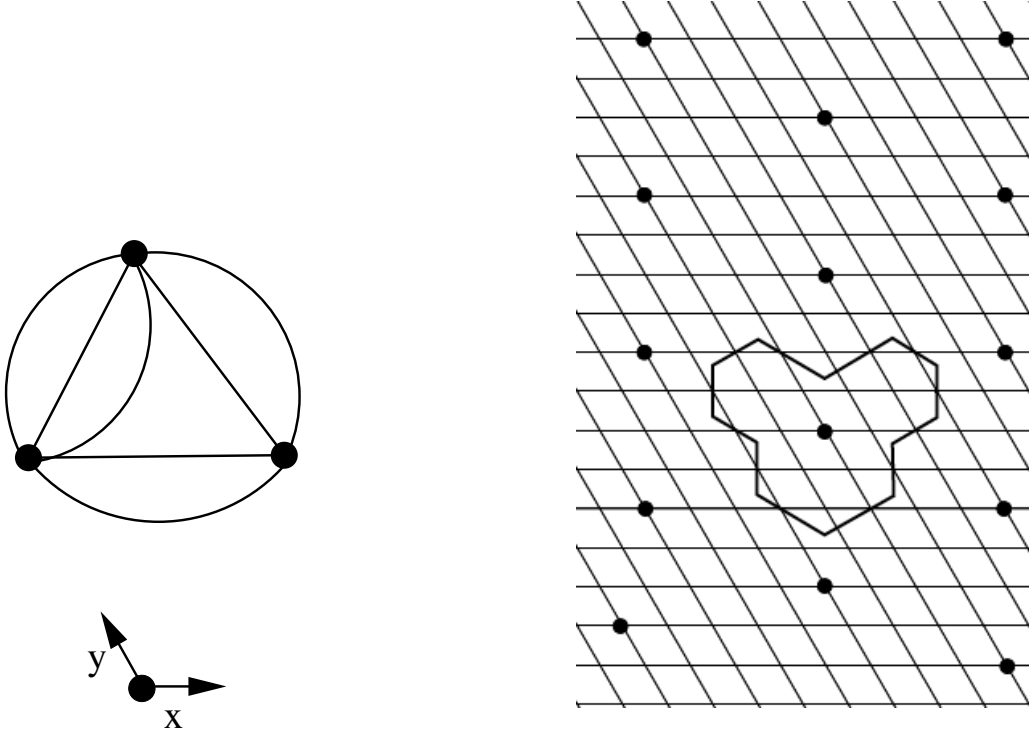


Figure 1: The shape of a Voronoi-cell in the Laplacian lattice of a graph with three vertices. The multi-graph G has three vertices and 7 edges. The lattice A_2 is generated by the two vectors $x = (1, -1, 0)$ and $y = (-1, 0, 1)$. The corresponding Laplacian sublattice of A_2 , whose elements are denoted by \bullet , is generated by the vectors $(-5, 3, 2) = -3x + 2y$ and $(3, -5, 2) = 5x + 2y$ (and $(2, 2, -4) = -2x - 4y$), which correspond to the vertices of G . The Delaunay triangulation consists of all translates of the triangle with vertices $(0, 0, 0)$, $(-5, 3, 2)$ and $(-2, -2, 4)$ and the triangle with vertices $(0, 0, 0)$, $(3, -5, 2)$ and $(-2, -2, 4)$ by points in the Laplacian lattice of G .

2.4.2 Distance function induced by a Discrete Point Set

Given a polyhedral distance function \mathcal{P} and a discrete point set S , we define a function $h_{\mathcal{P},S} : \mathbb{R}^n \rightarrow \mathbb{R}$ as:

$$h_{\mathcal{P},S}(p) = \min_{q \in S} \{d_{\mathcal{P}}(p, q)\} \quad (4)$$

In particular, the notion of local minima and local maxima of the distance function $h_{\mathcal{P},S}$ turns out to be useful:

Definition 10. (Local Maxima and Local Minima of $h_{\mathcal{P},S}$) Let $B(p, \epsilon)$ be the Euclidean ball of radius ϵ centered at p . A point c in \mathbb{R}^n is called a local minimum of $h_{\mathcal{P},S}$ if there exists an $\epsilon > 0$ such that $h_{\mathcal{P},S}(c) \leq h_{\mathcal{P},S}(q)$ for all $q \in B(c, \epsilon)$. A point c in \mathbb{R}^n is called a local maximum of $h_{\mathcal{P},S}$ if there exists an $\epsilon > 0$ such that $h_{\mathcal{P},S}(c) \geq h_{\mathcal{P},S}(q)$ for all $q \in B(c, \epsilon)$.

We have the following characterization of local minima of $h_{\mathcal{P},S}$:

Lemma 12. A point $q \in \mathbb{R}^n$ is a local minimum of $h_{\mathcal{P},S}$ if and only if $q \in S$.

We denote the set of local maxima of $h_{\mathcal{P},S}$ by $\text{Crit}_{\mathcal{P}}(S)$. The characterization of the local maximum of $h_{\mathcal{P},S}$ turns out to be more complex. In general, the local maxima of $h_{\mathcal{P},S}$ are a

subset of Voronoi vertices of S with respect of $-\mathcal{P}$. We will see in the next section that the local maxima of the Laplacian lattice with respect to \triangle are essentially the acyclic orientations of G .

2.4.3 Delaunay Triangulations

We define the Delaunay triangulation of a point set under a polyhedral distance function as follows:

Definition 11. (Delaunay Triangulation under a Polyhedral Distance Function) *A triangulation \mathcal{T} of a discrete point set S in \mathbb{R}^d is a Delaunay triangulation of S under the polyhedral distance function d_S if for every point c in $\text{Crit}_{\mathcal{P}}(S)$ there exists a simplex K in \mathcal{T} such that $Q(c, h_{\mathcal{P}, S}(c))$ contains the vertices of K in its boundary.*

The Delaunay triangulation under the simplicial distance function d_{\triangle} is closely related to the notion of Scarf complex associated with a lattice [22]. In fact, in the case of multigraphs whose Laplacian lattice has no zero entries, the Scarf complex coincides with the Delaunay triangulation, but in general the Scarf complex is a subcomplex of the Delaunay triangulation. The Scarf complex of a lattice was first considered in the context of mathematical economics and integer programming, and was later used in commutative algebra to determine free resolutions of the associated lattice ideal. The author is indebted to Bernd Sturmfels for pointing out this connection to him.

Remark 4. *Note that Definition 11 is a natural extension of the notion of Delaunay triangulation under the Euclidean distance. The Delaunay triangulation under the Euclidean distance has the property that every Voronoi vertex is the circumcenter of some Delaunay simplex. See a book on discrete geometry, for example [5] for more details.*

In the following sections, we undertake a detailed study of the Laplacian lattice under the simplicial distance function d_{\triangle} . Though we do not always mention it explicitly we always assume that the underlying distance is the simplicial distance function d_{\triangle} .

3 Voronoi Diagram and Delaunay triangulation of the Laplacian lattice under the simplicial distance function

In this section, we describe the Voronoi diagram, Delaunay triangulation and the local maxima of the simplicial distance function induced by the distance function d_{\triangle} on the Laplacian lattice. We draw heavily from Section 6 of [2]. This section is intended to serve two purposes, it contributes to providing a complete description of the geometry of the Laplacian lattice under the simplicial distance function d_{\triangle} and secondly, the results in this section are frequently used in the rest of the paper.

Let $\{b_0, \dots, b_n\}$ be the rows of the Laplacian matrix of G . For each permutation $\sigma \in S_{n+1}$, where S_{n+1} is the symmetric group on $n+1$ -elements. we define

$$u_i^{\sigma} = \sum_{j=0}^{i-1} b_{\sigma(j)} \quad (5)$$

for i from 0 to n . Note that $u_n^{\pi} = 0$ for every permutation $\pi \in S_{n+1}$.

3.1 Local Maxima of the simplicial distance function

Given a permutation π on the $n + 1$ vertices of G , define the ordering $\pi(v_0) <_\pi \pi(v_1) <_\pi \dots <_\pi \pi(v_n)$ and orient the edges of graph G according to the ordering defined by π i.e. there is an oriented edge from v_i to v_j if $(v_i, v_j) \in E$ and if $v_i <_\pi v_j$ in the ordering defined by π . Consider the acyclic orientation induced by a permutation π on the set of vertices of G and define $\nu_\pi = (\text{indeg}_\pi(v_0), \dots, \text{indeg}_\pi(v_n))$, where $\text{indeg}_\pi(v)$ is the indegree of the vertex v in the directed graph oriented according to π . Define $\text{Ext}^c(L_G) = \{\nu_\pi + q \mid \pi \in S_{n+1}, q \in L_G\}$.

Theorem 3. (Theorem 6.1 of [2]) *The elements of the local maxima of the simplicial distance function h_{Δ, L_G} are precisely the orthogonal projections of the elements of $\text{Ext}^c(L_G)$ onto H_0 .*

3.2 Voronoi Diagram

We shall first consider the case of Laplacian lattices generated by multigraphs where every pair of vertices are connected by an edge (for example, the complete graph). As we shall see, these graphs are technically easier to handle than the general case. Recall that distinct lattice points p, q are Voronoi neighbours if the intersection of their Voronoi cells are non-empty.

Theorem 4. (Corollary 6.18, [2]) *For the Laplacian lattice of a graph for which every pair of vertices are connected by an edge, a lattice point q is a Voronoi neighbour of the origin with respect to d_Δ if and only if q is of the form u_i^σ for some $\sigma \in S_{n+1}$ and an integer i from 1 to $n - 1$.*

The case of general multigraphs is slightly more involved and are a consequence of results of [2]. Since Theorem 5 is not used in the rest of the paper, a reader may skip the theorem for the first reading.

Theorem 5. *The Voronoi neighbours of the origin with respect to d_Δ are precisely the set of non-zero lattice points that are contained in $\bar{\Delta}(c_\pi, h_{\Delta, L_G}(c_\pi))$ where c_π is the orthogonal projection of ν_π onto H_0 and $\bar{\Delta}(c_\pi, h_{\Delta, L_G}(c_\pi))$ is the simplex $h_{\Delta, L_G}(c_\pi) \cdot \bar{\Delta} + c_\pi$ (see Definition 6).*

Proof. First, we know that the Voronoi cell of every non-zero lattice point q in $\bar{\Delta}(c_\pi, h_{\Delta, L_G}(c_\pi))$ shares c_π with the Voronoi cell of the origin and hence q is a Voronoi neighbour of the origin. Conversely, consider a lattice point q that is a Voronoi neighbour of the origin O we know that the simplices centered at q and O share a point m say in the boundary of the arrangement of simplices Δ centered at lattice points and the radius of simplices is r for some real number $r > 0$. By the duality theorem (Theorem 8.3 of [2]), we know that there is a point c in $\text{Crit}_\Delta(L_G)$ such that $\bar{\Delta}(c, \text{Cov}_\Delta(L_G) - r)$ contains m . Applying triangle inequality, we deduce that $d_{\bar{\Delta}}(c, O) \leq \text{Cov}_\Delta(L_G)$ but we also know that $d_{\bar{\Delta}}(c, q') \geq \text{Cov}_\Delta(L_G)$ for all $q' \in L_G$. Hence, $d_{\bar{\Delta}}(c, O) = d_\Delta(O, c) = \text{Cov}_\Delta(L_G)$. Using item ii of Theorem 6.9 of [2] we know that $c = c_\pi$ for some permutation $\pi \in S_{n+1}$. Similarly, we also know that $d_{\bar{\Delta}}(c, q) = \text{Cov}_\Delta(L_G)$ and hence, q is contained in $\bar{\Delta}(c_\pi, h_{\Delta, L_G}(c_\pi))$. \square

3.3 A perturbation trick

For the case of graphs where every pair of vertices are connected by an edge, Theorem 4 gives a useful characterization of the Voronoi neighbours of the origin, but for the case of general connected graphs, the characterization obtained in Theorem 5 is not explicit enough for our purposes. In the subsequent sections, we use the following perturbation trick to handle the

Laplacian lattice of a general connected graph: we perturb the Laplacian lattice and scale it to the “nice” case, i.e. to the case of lattices generated by graphs where every pair of vertices are connected by an edge and study the limit as the perturbation tends to zero. More precisely, we consider lattices generated by the following perturbed basis: we add a rational number $\epsilon > 0$ to every non-diagonal element b_{ij} of the Laplacian matrix and then set the diagonal elements so that the row sum and column sum is zero. We call such a perturbation a **standard** perturbation and denote the vector obtained by perturbing b_i by b_i^ϵ . The following lemma characterizes the Voronoi neighbours of the perturbed Laplacian lattice:

Lemma 13. *Let L_G be the Laplacian lattice of the graph and L_G^ϵ be the lattice obtained by perturbing L_G according to the standard perturbation. The Voronoi neighbours of L^ϵ under the distance function d_Δ are of the form $u_S^\epsilon = b_{i_1}^\epsilon + b_{i_2}^\epsilon + \dots + b_{i_k}^\epsilon$ where i_j s are distinct.*

Proof. Since the perturbation ϵ is rational, we can scale the lattice by a factor λ , say to obtain the Laplacian lattice of a graph where every pair of vertices is connected by an edge. We can apply Theorem 4 to the scaled lattice λL^ϵ to deduce that the Voronoi neighbours of the lattice λL^ϵ are of the form λu_S^ϵ for some subset S . Finally, observe that the Voronoi neighbours of a lattice are preserved under scaling to complete the proof. \square

Remark 5. *Note that there are different ways of perturbing the basis to obtain a “nice” lattice. For example, we can add an $\epsilon > 0$ only to the non-diagonal elements that are zero and then set the diagonal elements such that the row sum and column sum is zero.*

3.4 Delaunay Triangulation

Theorem 6. *(See Section 6.1 of [2]) Let S^σ be the convex hull of $u_0^\sigma, \dots, u_n^\sigma$. The set $\{S^\sigma + p\}_{\sigma \in S_{n+1}, p \in L_G}$ is a Delaunay triangulation of L_G and is a unique Delaunay triangulation if G is a graph that contains the complete graph.*

We shall see in Section 7 that the the number of different graphs that have L_G as their Laplacian lattice upper bounds the number of “different” Delaunay triangulations of L_G .

4 Packing and Covering Radius of the Laplacian lattice

We will show that the packing radius of the graph under the distance function Δ (and $\bar{\Delta}$) is essentially (up to a factor depending on the number of vertices) the minimum cut of a graph.

Definition 12. (ℓ_1 -Minimum Cut) *For a non-trivial cut S of $V(G)$ i.e. S is neither empty nor equal to $V(G)$, define the weight of the cut $\mu_1(S) = \sum_{v \in S} \deg_{S, \bar{S}}(v)$ where $\deg_{S, \bar{S}}(v)$ is the degree of the vertex v across the cut S . Now define the ℓ_1 -minimum cut $MC_1(G)$ as the minimum of $\mu_1(S)$ over all non-trivial cuts S .*

Remark that ℓ_1 -minimum cut of a graph is the same as the minimum cut of a graph. We call it the ℓ_1 -minimum cut to distinguish from a variant the “ ℓ_∞ -minimum cut” that we will encounter in the next section.

For points $p, q \in \mathbb{R}^{n+1}$, the max-sum $p \oplus q$ is defined as $(\max(p_1, q_1), \dots, \max(p_{n+1}, q_{n+1}))$.

Lemma 14. *For any point p in H_0 , Δ -midpoint m of p and the origin O is the projection of the max-sum of the two points onto H_0 and $d_\Delta(p, m) = d_\Delta(O, m) = \|p \oplus O\|_1 / (n + 1)$.*

Proof. For a point $r \in \mathbb{R}^{n+1}$ let H_r^+ be the domination cone defined as: $H_r^+ = \{r' \mid r' \in \mathbb{R}^{n+1}, r' - r \geq O\}$. Consider H_p^+ and H_O^+ and observe that the point with minimum ℓ -norm in their intersection is $p \oplus O$. Project the system onto H_0 along the normal $(1, \dots, 1)$. A simple computation shows that the projection of the cones onto H_0 are simplices and these simplices are dilated and translated copies of Δ and the only point of intersection of the simplices is the projection of the max-sum. Hence, we obtain $d_\Delta(O, m) = d_\Delta(p, m) = \|p \oplus O\|_1 / (n+1)$. \square

Lemma 15. *For a subset S of the rows of the Laplacian matrix of the graph, let $u_S = \sum_{b_i \in S} b_i$. The ℓ_1 -norm of the max-sum of u_S and O is the size of the cut defined by S .*

Proof. The max-sum $u_S \oplus O$ of u_S and O is given by:

$$(u_S \oplus O)_i = \begin{cases} u_{Si}, & \text{if } (u_S)_i > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

Hence, we consider the positive coordinates of u_S . Now note that since the sum of coordinates of u_S is zero, the absolute sum of the positive coordinates is equal to absolute sum of negative coordinates. Furthermore, the negative coordinates are characterized by vertices that do not belong to the cut and that the sum of the absolute values of the negative-valued coordinates of u_S is the size of the cut S . \square

Theorem 7. *The packing radius of L_G under the simplicial distance function d_Δ is equal to $\frac{MC_1(G)}{n+1}$ where $MC_1(G)$ is the size of the ℓ_1 -minimum cut.*

Proof. First, observe that the lattice point that defines the packing radius under the distance function d_Δ is a Voronoi neighbour of the origin under d_Δ . Hence, we only restrict to the Voronoi neighbours. Now for the case of general connected graphs, the characterization of the Voronoi neighbours of the origin obtained in Theorem 5 is not explicit enough. Hence, we perform the standard perturbation of the Laplacian matrix (see Subsection 3.2) and consider lattices generated by these perturbed matrices. Using Lemma 13, we know that the packing radius of the perturbed lattice L_G^ϵ is defined by a point of the form $u_S = \sum_{i \in S} b_i^\epsilon$ for some non-trivial subset S of $V(G)$ and b_i^ϵ being the perturbed vector of b_i . Using the fact that the packing radius is preserved under perturbation (see Appendix Subsection 9.1), we deduce that the packing radius is defined by a point of the form u_S for a non-trivial subset S of $V(G)$. By Lemma 14 and the definition of packing radius we have $\text{Pac}_\Delta(L_G) = \min_S \|u_S \oplus O\|_1 / n + 1$ where the minimum is taken over all the non-trivial cuts S . We now use Lemma 15 to deduce that $\min_S \|u_S \oplus O\|_1$ is equal to the size of the minimum cut of G . \square

Theorem 8. (Section 8.3 of [2]) *The covering radius of the Laplacian lattice is equal to $\frac{g+n}{n+1}$.*

5 The Shortest Vector of the Laplacian lattice

We will provide a combinatorial interpretation of the norm of the shortest vector of the Laplacian lattice under the simplicial distance function d_Δ . We will see that the norm of the shortest vector of the Laplacian lattice under the simplicial distance function d_Δ is in fact a certain variant of the minimum cut in the graph. A precise definition follows:

Definition 13. (ℓ_∞ -Minimum Cut) *For a non-trivial cut S of $V(G)$ i.e. S is neither empty nor equal to $V(G)$, define the weight of the cut as $\mu_\infty(S) = \max\{\deg_{S, \bar{S}}(v) \mid v \in \bar{S}\}$ where $\deg_{S, \bar{S}}(v)$ is the degree of the vertex v across the cut S . Now define the ℓ_∞ -minimum cut $MC_\infty(G)$ as the minimum of $\mu_\infty(S)$ over all non-trivial cuts S .*

Note that for a simple connected graph G , we have $\text{MC}_\infty(G) = 1$.

Theorem 9. *The length of the shortest vector $\nu_\Delta(L_G)$ of the Laplacian lattice under the simplicial distance function d_Δ is equal to $\text{MC}_\infty(G)$.*

Proof. First let us consider the case where G is a multigraph where every pair of vertices are connected by an edge. By Lemma 11, we know that every shortest vector in the distance function d_Δ must be a Voronoi neighbour of the origin under d_Δ . We know that the Voronoi neighbours of the origin under d_Δ are of the form: $u_S = \sum_{i \in S} b_i$ for some non-trivial subset S of $V(G)$. For the case of general connected graphs, the characterization of the Voronoi neighbours of the origin obtained in Theorem 5 is not explicit enough. Hence, we perform the standard perturbation (see Subsection 3.2) of the Laplacian matrix and consider lattices generated by these perturbed matrices. By Lemma 13, we know that the shortest vector of the perturbed lattice L_G^ϵ is defined by a point of the form $u_S = \sum_{i \in S} b_i^\epsilon$ for some non-trivial subset S of $V(G)$ and b_i^ϵ being the perturbed vector of b_i . Using the fact that the quantity $\nu_\Delta(\cdot)$ is preserved as the perturbation tends to zero (See Appendix Section 9.1), we deduce that the shortest vector for the general lattice must be of the form: $u_S = \sum_{i \in S} b_i$ for some non-trivial subset S of $V(G)$.

Consider a subset S such that u_S is a shortest vector. First, recall that $d_\Delta(O, u_S) = |\min_j u_{Sj}|$. Now, since $u_S \neq O$ only the negative coordinates of u_S define $d_\Delta(O, u_S)$ and the negative coordinates are indices j such that $v_j \notin S$. Hence $d_\Delta(O, u_S) = \max\{\deg_{S, \bar{S}}(v) \mid v \in \bar{S}\}$. We have:

$$\nu_\Delta(L_G) = \min_S \{d_\Delta(O, u_S)\} = \min_S \max\{\deg_{S, \bar{S}}(v) \mid v \in \bar{S}\} = \text{MC}_\infty(G). \quad (7)$$

□

Corollary 2. *For a simple connected graph G , we have $\nu_\Delta(L_G) = 1$.*

We will now apply the combinatorial interpretation of the parameters of the Laplacian lattice that we have gained in the previous sections to answer some natural questions that arise from the correspondence between the graph and its Laplacian lattice.

6 The Delaunay polytope of the Laplacian lattice characterizes the graph completely up to isomorphism

A natural question that arises with the correspondence between the Laplacian lattice and a graph is whether the Laplacian lattice characterizes the underlying graph completely up to isomorphism? The first observation towards answering this question is the following lemma:

Lemma 16. *The Laplacian lattice of any tree on $n + 1$ vertices is the root lattice A_n .*

Proof. We know that the Laplacian lattice is a sublattice of the root lattice A_n and the covolume of a Laplacian lattice with respect to A_n is equal to the number of spanning trees of the graph. This implies that in the case of trees the covolume of the Laplacian lattice is equal to one. Hence, the Laplacian lattice of a tree is the root lattice A_n itself. □

Lemma 16 shows that the Laplacian lattice itself does not characterize a graph completely up to isomorphism. However, we will now see that the Delaunay triangulations of the Laplacian lattice L with respect to the simplicial distance function provide more refined information

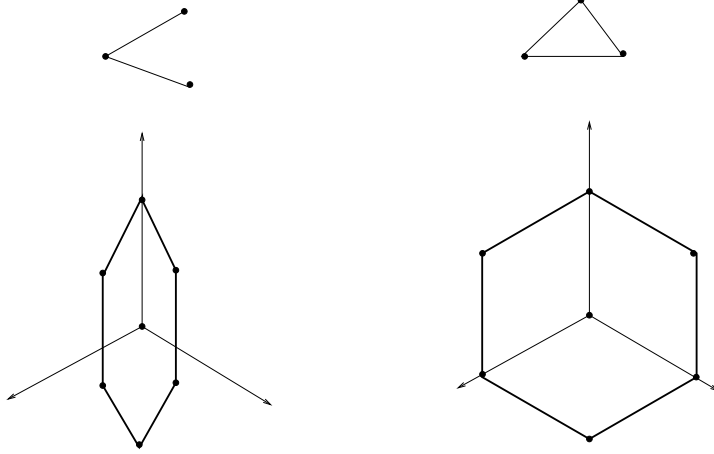


Figure 2: The Delaunay polytope of a path on two edges (left) and a triangle.

about the graphs that have L as their Laplacian lattice. More precisely, each graph provides a Delaunay triangulation of its Laplacian lattice and the Delaunay polytope of the origin characterizes the graph completely up to isomorphism. In the course of showing this result, we also study the structure of the Delaunay polytope. As we noted, in the case of graphs with no zero entries in the Laplacian matrix our study of the Delaunay triangulation is in fact a study of the Scarf complex of the Laplacian lattice and in general, the Delaunay triangulation contains the Scarf complex of L_G .

6.1 The structure of the polytope $H_{Del_G}(O)$

We know from Theorem 6 that each graph provides a Delaunay triangulation of its Laplacian lattice in the following manner: Recall that $\{b_0, \dots, b_n\}$ are the rows of the Laplacian matrix of G . For each permutation $\sigma \in S_{n+1}$, we define $u_i^\sigma = \sum_{j=0}^i b_{\sigma(j)}$ for i from 0 to n . Note that $u_n^\pi = (0, \dots, 0)$ for every permutation $\pi \in S_{n+1}$. We define the simplex Δ_σ as the convex hull of $u_0^\sigma, \dots, u_n^\sigma$. The Delaunay polytope of the origin i.e. the set of Delaunay simplices with the origin as a vertex, is given by $\{\Delta_\sigma\}_{\sigma \in S_{n+1}}$. See Figure 2 for the Delaunay polytope of some small graphs. We denote this polytope by $H_{Del_G}(O)$. We know from Theorem 6 that each graph provides a Delaunay triangulation of its Laplacian lattice in the following manner: Recall that $\{b_0, \dots, b_n\}$ are the rows of the Laplacian matrix of G . For each permutation $\sigma \in S_{n+1}$, we define $u_i^\sigma = \sum_{j=0}^i b_{\sigma(j)}$ for i from 0 to n . Note that $u_n^\pi = (0, \dots, 0)$ for every permutation $\pi \in S_{n+1}$. We define the simplex Δ_σ as the convex hull of $u_0^\sigma, \dots, u_n^\sigma$. The Delaunay polytope of the origin i.e. the set of Delaunay simplices with the origin as a vertex, is given by $\{\Delta_\sigma\}_{\sigma \in S_{n+1}}$. See Figure 2 for the Delaunay polytope of some small graphs. We denote this polytope by $H_{Del_G}(O)$.

We first describe the vertex set of $H_{Del_G}(O)$. A study of low-dimensional examples leads us to the claim that every vertex of $H_{Del_G}(O)$ is of the form u_k^σ for some integer k from 0 to $n-1$ and a permutation $\sigma \in S_{n+1}$. In order to show this claim, we proceed as follows: we consider the convex hull, call it $H'(G)$, of points u_k^σ for k from 0 to n and over all permutations $\sigma \in S_{n+1}$. We then show that every point of the form u_k^σ for k from 0 to $n-1$ is a vertex of $H'(G)$ and that $H'(G) = H_{Del_G}(O)$.

Lemma 17. *Every point of the form u_k^σ is a vertex of $H'(G)$ where k varies from 0 to $n-1$ and $\sigma \in S_{n+1}$.*

Proof. We show that for every point of the form $u_k^\sigma = \sum_{i=0}^k b_\sigma(i)$ for k from 0 to $n-1$ there exists a point w such that the linear functional: $f(x) = w \cdot x^t$ has the property:

$$f(b_{\sigma(i)}) = \begin{cases} > 0 & \text{if } 0 \leq i \leq k, \\ < 0 & \text{otherwise.} \end{cases}$$

The details of the construction of the functional f are as follows: Consider the set $C_{\sigma,k}$ of points $(p_0, \dots, p_n) \in H_0$ such that

$$p_{\sigma(i)} = \begin{cases} > 0 & \text{if } 0 \leq i \leq k, \\ < 0 & \text{otherwise.} \end{cases}$$

We make the following observations:

1. $C_{\sigma,k}$ is a cone.
2. $C_{\sigma,k}$ is not empty for k from 0 to $n-1$.

Take a point q in $C_{\sigma,k}$. Since G is connected, L_G is a lattice of dimension n and a basis of L_G spans H_0 . Consider the basis $\{b_0, \dots, b_{n-1}\}$ (the first n rows of $Q(G)$). Let $q = \sum_{i=0}^{n-1} v_i \cdot b_i$. Set $w_i = v_i$ for i from 0 to $n-1$ and $w_n = 0$. Using the symmetry of the Laplacian matrix, we can easily verify that the functional f has the desired properties. By the properties of f , it follows that u_k^σ is the unique maximum of $f(x)$ among vertices of the form u_j^π for an arbitrary permutation $\pi \in S_{n+1}$.

This implies that u_k^σ is also the unique maximum of $f(x)$ over the polytope $H'(G)$. Using standard arguments in linear optimization [24], this implies that u_k^σ is a vertex of $H'(G)$. \square

Next we characterize points in the simplex Δ_σ :

Lemma 18. *A point p is contained in Δ_σ if and only if p can be written as $p = \sum_{i=0}^n \lambda_i b_{\sigma(i)}$ where $1 = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0$.*

Proof. If p is contained in Δ_σ then, we can write: $p = \sum_{i=0}^n \mu_i u_i^\sigma$ where $\mu_i \geq 0$ and $\sum_{i=0}^n \mu_i = 1$. We can plug in $u_k^\sigma = \sum_{i=0}^k b_{\sigma(i)}$ to the equation to obtain: $p = \sum_{i=0}^n \lambda_i b_{\sigma(i)}$ where $\lambda_k = \sum_{i=k}^n \mu_i$. Now, observe that $1 = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0$.

Conversely, if a point can be written as $p = \sum_{i=0}^n \lambda_i b_{\sigma(i)}$ where $1 = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0$. We set $\mu_i = \lambda_i - \lambda_{i+1}$ for $0 \leq i \leq n-1$ and $\mu_n = \lambda_n$ and we have $p = \sum_{i=0}^n \mu_i u_i^\sigma$. We finally verify the following properties: i. $\mu_i \geq 0$, since $\lambda_i \geq \lambda_{i+1}$ for i from 0 to $n-1$, ii. $\lambda_n \geq 0$ and iii. $\sum_{i=0}^n \mu_i = \lambda_0 = 1$. This shows that p is contained in Δ_σ . \square

Corollary 3. *The set $H_{Del_{G_1}}(O)$ is convex.*

Proof. Consider points p_1 and p_2 in simplices Δ_{σ_1} and Δ_{σ_2} for some $\sigma_1, \sigma_2 \in S_{n+1}$. By Lemma 18, we can write: $p_1 = \sum_{i=0}^n \lambda_i^1 b_{\sigma_1(i)}$ and $p_2 = \sum_{i=0}^n \lambda_i^2 b_{\sigma_2(i)}$ where $1 = \lambda_0^i \geq \lambda_1^i \geq \dots \geq \lambda_n^i \geq 0$ for $i \in \{1, 2\}$. Each point p in the line segment joining p_1 and p_2 can be written as $\mu p_1 + (1-\mu)p_2$ for some $0 \leq \mu \leq 1$. We can write: $p = \sum_{i=0}^n \lambda_i b_i$ and there exists a permutation σ such that $1 \geq \lambda_{\sigma(0)} \geq \lambda_{\sigma(2)} \geq \dots \geq \lambda_{\sigma(n)} \geq 0$. We add $O = (1 - \lambda_{\sigma(0)}) \sum_{i=0}^n b_i$ to the right hand side of the equation to obtain a point of the form stated in Lemma 18. This concludes the proof. \square

Lemma 19. *For any undirected connected graph G , we have $H'(G) = H_{Del_G}(O)$.*

Proof. Consider a point p in $H_{Del_G}(O)$. By definition, p belongs to some simplex of the form Δ_σ . Now, observe that all the vertices of Δ_σ except the origin are contained in $H'(G)$. Now, let $p = \sum_{i=0}^{n-1} \lambda_i u_i^\sigma + \lambda_n \cdot O$ for some $\lambda_i \geq 0$ and $\sum_{i=0}^n \lambda_i = 1$. We have $p = \sum_{i=0}^{n-1} \lambda_i u_i^\sigma + \lambda_n \cdot O =$

$\sum_{i=0}^{n-1} \lambda_i u_i^\sigma + \lambda_n(b_0 + \sum_{j=1}^n b_j)/2$. This shows that p can be written as a convex combination of points in $H'(G)$ and hence p is contained in $H'(G)$.

We show the converse by contradiction. Assume that there exists a point p in $H'(G)/H_{Del_G}(O)$. By Corollary 3, $H_{Del_G}(O)$ is a convex polytope and hence, a closed subset of H_0 equipped with the Euclidean topology. By the separation theorem for closed convex sets [23], there exists a linear functional $f(x)$ such that $f(p) < 0$ and $f(y) > 0$ for all $y \in H_{Del_G}(O)$. Since u_k^σ is contained in $H_{Del_G}(O)$, we know that $f(u_k^\sigma) > 0$ for all $0 \leq k \leq n$ and $\sigma \in S_{n+1}$. But since, p is contained in $H'(G)$ it can be written as a convex combination of the vertices of $H'(G)$ and the vertices of $H'(G)$ are points of the form u_k^σ for k from 0 to $n-1$ and hence $f(p) > 0$. We obtain a contradiction. \square

Corollary 4. *The set $H_{Del_G}(O)$ is a convex polytope and has $2^{n+1} - 2$ vertices.*

We now describe the facet structure of $H_{Del_G}(O)$. We know that every vertex of $H_{Del_G}(O)$ is of the form:

$$v = \sum_{j=0}^k b_{i_j}, \text{ for } k \text{ from } 1 \text{ to } n \text{ and } b_{i_j} \text{ s are all distinct.} \quad (8)$$

We define the set V_i , for i from 0 to n as the subset of vertices that contain b_i in their representation of the form stated in (8). Define the set $F_{i,j} = V_i - V_j$.

Lemma 20. *For each integer $0 \leq i, j \leq n$ and $i \neq j$, the affine hull of the elements of $F_{i,j}$ is a $n-1$ -dimensional affine space.*

Proof. For the sake of convenience, we consider $F_{0,n}$. The set $S = \{b_0, b_0 + b_1, \dots, b_0 + b_1 + b_2 + \dots + b_{n-1}\}$ is contained in $F_{0,n}$. Since G is connected, the elements of S are linearly independent and hence their affine hull is a $n-1$ -dimensional affine space.

Now, we show that every element of $F_{0,n}$ is contained in the affine hull of S . The affine hull of S consists of elements of the form $\sum_{i=0}^n \alpha_i u_i$ where $u_i = \sum_{k=0}^i b_k$ and $\sum_{i=0}^n \alpha_i = 1$. This can be written as $\sum_{i=0}^n \lambda_i b_i$ where $\lambda_i = \sum_{j=i}^n \alpha_j$. This means that $\lambda = L\alpha$ where $\lambda = (\lambda_0, \dots, \lambda_n)$, $\alpha = (\alpha_0, \dots, \alpha_n)$ and L is the lower triangular matrix with unit entries. By definition, the elements of $F_{0,n}$ are of the form: $b_0 + b_{i_1} + \dots + b_{i_k}$ where $i_j \notin \{0, n\}$. Now consider an element $u = b_0 + b_{j_1} + \dots + b_{j_k}$ in $F_{0,n+1}$ and let $S' = \{0, j_1, \dots, j_k\}$. Set $\lambda' = i_{S'}$, the indicator vector of the set S' and since L is invertible there exists a unique vector $\alpha' = L^{-1}\lambda'$ and we verify that $\sum_{i=0}^n \alpha'_i$ is the coefficient of b_0 in u and is hence equal to 1. This shows that u is contained in the affine hull of S . The same argument can be done for any $F_{i,j}$ by replacing 0 by i and n by j . \square

Lemma 21. *For each integer $0 \leq i, j \leq n$ and $i \neq j$, the convex hull of the elements of $F_{i,j}$ is a facet of $H_{Del_G}(O)$. Furthermore, each element of $F_{i,j}$ is in convex position.*

Proof. For the sake of convenience, we consider $F_{0,n}$. By Lemma 20, the affine hull of the elements of $F_{0,n}$ spans a $n-1$ -dimensional space K and we know that $H_{Del_G}(O)$ is a n -dimensional polytope. In order to show that $F_{0,n}$ is a facet of $H_{Del_G}(O)$ we need to show that the affine hull of the elements of $F_{0,n}$ supports $H_{Del_G}(O)$. This can be seen as follows: since G is connected we know that $\{b_0, \dots, b_{n-1}\}$ is a basis of H_0 . Hence, every point in H_0 can be uniquely written as $\sum_{i=0}^{n-1} \alpha_i b_i$ for some $\alpha_i \in \mathbb{R}$ and the hyperplane K is given by $\alpha_0 = 1$. Now, consider a vertex v of $H_{Del_G}(O)$ not contained in $F_{0,n}$. Let $v = \sum_{j=1}^k b_{i_j}$. Since v is not contained in $F_{0,n}$, either b_0 is not contained in the sum representing v or b_n is contained in the

sum. If b_0 is not contained in the sum and b_n is contained in the sum then $\alpha_0 = -1$. If b_0 and b_n are contained in the sum then $\alpha_1 = 0$ and also, if b_0 and b_n are not contained in the sum then $\alpha_0 = 0$. This shows that all the vertices of $H_{Del_G}(O)$ that are not in $F_{0,n-1}$ are strictly contained in K^- i.e. the halfspace $\alpha_0 \leq 1$. This suffices to conclude that $K \cap H_{Del_G}(O) = F_{0,n}$. By Lemma 17, all the elements of $F_{0,n}$ are in convex position and that concludes the proof. \square

Lemma 22. *The facets of $H_{Del_G}(O)$ are exactly of the form $F_{i,j}$ for $0 \leq i, j \leq n$ and $i \neq j$.*

Proof. By construction, the facets of $H_{Del_G}(O)$ must be contained in the affine hull of the facets of Δ_σ . By Lemma 21, we know that the affine hull of any facet of Δ_σ not containing the origin contains a facet of $H_{Del_G}(O)$. Now, it suffices to show that the affine hull of a facet of Δ_σ containing the origin does not contain a facet of $H_{Del_G}(O)$. Consider a facet F of Δ_σ . We may assume, without loss of generality, that F contains all vertices of Δ_σ apart from u_k^σ for some $0 \leq k \leq n-1$. Assume that F is a facet of $H_{Del_G}(O)$. This means that there is an affine function f such that

P1. $f(u_i^\sigma) = c$ for all $i \neq k$.

P2. $f(v) > c$ for all vertices of $H_{Del_G}(O)$ that are not contained in the affine hull of F .

Note that by the property P1, we have $f(O) = c$. Consider the linear function $g(x) = f(x) - f(O)$. We have $g(u_i^\sigma) = 0$ for all $i \neq k$. Now, suppose that $g(u_k^\sigma) = 0$, then $f(u_k^\sigma) = f(u_j^\sigma)$ for all $j \neq k$, but since G is connected, u_k^σ is not contained in the affine hull of F and this contradicts the property P2. If $g(u_{\sigma(k)}) < 0$, then $f(u_k^\sigma) < c$ and hence again contradicts property P2. On the other hand, if $g(u_{\sigma(k)}) > 0$, then $g(-u_{\sigma(k)}) < 0$ and $f(-u_k^\sigma) < c$ and $-u_k^\sigma$ is indeed a vertex of $H_{Del_G}(O)$. This again contradicts the property P2. This concludes the proof. \square

Corollary 5. *The number of facets of $H_{Del_G}(O)$ is $n \cdot (n+1)$.*

We now characterize the edges of $H_{Del_G}(O)$.

Lemma 23. *The edges of $H_{Del_G}(O)$ are of the form: $e_k^\sigma = (u_k^\sigma, u_{k+1}^\sigma)$ for k from 0 to $n-1$ and $\sigma \in S_{n+1}$.*

Proof. The proof is modelled similarly as the proof of Lemma 17. We show that for each e_k^σ , there exists a linear function $f(x) = w \cdot x^t$ such that $f(u_k^\sigma) = f(u_{k+1}^\sigma) = c$ and $f(v) < c$ for all other vertices v of $H_{Del_G}(O)$.

Consider the set $C_{e_k^\sigma}$ of points $(p_1, \dots, p_{n+1}) \in H_0$ such that

$$p_{\sigma(i)} = \begin{cases} > 0 & \text{if } 0 \leq i \leq k, \\ = 0 & \text{if } i = k+1, \\ < 0, & \text{otherwise.} \end{cases}$$

We make the following observations:

1. $C_{e_k^\sigma}$ is a cone.
2. $C_{e_k^\sigma}$ for $0 \leq k \leq n-1$ is not empty.

Take a point q in $C_{\sigma,k}$. Since G is connected, L_G is a lattice of dimension n and a basis of L_G spans H_0 . Consider the basis $\{b_0, \dots, b_{n-1}\}$ (the first n rows of $Q(G)$) and write $q = \sum_{i=0}^{n-1} v_i \cdot b_i$. Set $w_i = v_i$ for i from 0 to $n-1$ and $w_n = 0$.

The functional $f(x) = w \cdot x^t$ attains its maximum precisely on the edge e_k^σ . Using standard arguments in linear optimization [24], this implies that e_k^σ is an edge of $H'(G)$.

Now it remains to show that there are no other edges of $H_{Del_G}(O)$. By construction of $H_{Del_G}(O)$, an edge of $H_{Del_G}(O)$ is contained on the affine hull of an edge of Δ^σ for some $\sigma \in S_{n+1}$. An edge of Δ^σ is of the form (u_i^σ, u_j^σ) for some $0 \leq i, j \leq n$ and $i \neq j$. We know that $(u_k^\sigma, u_{k+1}^\sigma)$ is an edge for $0 \leq k \leq n-1$ and $\sigma \in S_{n+1}$. Now, consider an edge $e_{i,j}$ of Δ^σ of the form (u_i^σ, u_j^σ) where $j \neq i+1$ and $i < j$. Assume that the affine hull of $e_{i,j}$ contains an edge of $H_{Del_G}(O)$. This implies that there exists an affine function f such that P1. $f(u_i^\sigma) = f(u_j^\sigma) = c$. P2. $f(v) \leq c$ for all other vertices of $H_{Del_G}(O)$ not contained in the affine hull of $e_{i,j}$. Let g be the linear function $g(x) = f(x) - f(O)$. Now, by property P1, we have $f(u_j^\sigma - u_i^\sigma) = g(u_j^\sigma - u_i^\sigma) = 0$. This means that $f(b_{\sigma(j)} + \dots + b_{\sigma(i+1)}) = g(b_{\sigma(j)} + \dots + g(b_{\sigma(i+1)})) = 0$. Now, either $g(b_{\sigma(j)}) = g(b_{\sigma(j-1)}) = \dots = g(b_{\sigma(i+1)}) = 0$ or there exists $j \leq k_1, k_2 \leq i+1$ such that $g(b_{\sigma(k_1)}) > 0$ and $g(b_{\sigma(k_2)}) < 0$. In the first case, we have $f(u_{i+1}^\sigma) = f(u_i^\sigma) + g(b_{\sigma(i+1)}) = c$ but by property P2, this means that u_{i+1}^σ is contained in the affine hull of $e_{i,j}$ and hence $u_{i+1}^\sigma = \lambda \cdot (u_i^\sigma - u_j^\sigma)$ for some $\lambda \in \mathbb{R}$. But this contradicts the connectivity of G . In the second case, we have $f(u_{\sigma(i)} + b_{\sigma(k_1)}) = f(u_i^\sigma) + g(b_{\sigma(k_1)}) > c$ and since $u_{\sigma(i)} + b_{\sigma(k_1)}$ is a vertex of $H_{Del_G}(O)$, we obtain a contradiction. This concludes the proof. \square

Problem 1. *We have characterized the zero, one and $n-1$ dimensional faces of $H_{Del_G}(O)$. Can we obtain similar characterizations of the other faces of $H_{Del_G}(O)$?*

6.2 Combinatorics of $H_{Del_G}(O)$.

Recall that the f -vector of an n -dimensional polytope P is the vector (f_0, \dots, f_{n-1}) where f_k is the number of k -dimensional faces of P . In the previous section, we showed that $f_0(H_{Del_G}(O)) = 2^{n+1} - 2$ and $f_{n-1}(H_{Del_G}(O)) = n \cdot (n+1)$.

For each vertex v , we denote by $d_k(v)$, the number of k -dimensional faces incident on v .

Lemma 24. *For a vertex v of the form $\sum_{j=0}^{k-1} b_{i_j}$ where the indices i_j s are all distinct, we have $d_{n-1}(v) = k \cdot (n+1-k)$.*

Proof. Consider a vertex v of the form $\sum_{j=0}^{k-1} b_{i_j}$ where i_j s are all distinct and let $S_v = \{i_0, \dots, i_{k-1}\}$. The facets that contain v are of the form: $F_{i,j}$ where $i \in S_v$ and $j \notin S_v$. There are $k \cdot (n+1-k)$ such facets. \square

Corollary 6. *Let $g(k) = k \cdot (n+1-k)$, if $g(k_1) = g(k_2)$ for $k_1 \neq k_2$ then $k_1 + k_2 = n+1$.*

Recall that the polytopes \mathcal{P}_1 and \mathcal{P}_2 in H_0 are congruent if there exists an isometry M such that $\mathcal{P}_2 = M(\mathcal{P}_1)$. By an isometry, we mean a map of the form $M \cdot x = A \cdot x + t$ where A is an orthogonal transformation and $t \in H_0$.

6.3 Correspondence between G and $H_{Del(G)}(O)$

For a convex polytope \mathcal{P} in H_0 , let $Aut(\mathcal{P})$ denote the automorphism group of \mathcal{P} . The following simple lemmas turn out to be useful.

Lemma 25. *Let \mathcal{P} be a polytope in \mathbb{R}^n , let M be an element of $Aut(\mathcal{P})$, then M permutes the vertices of \mathcal{P} .*

A simple calculation shows the following:

Lemma 26. *Every element of $Aut(\Delta)$ is an orthogonal transformation i.e. the translation part of the isometric map is zero.*

Theorem 10. *Let G_1 and G_2 be undirected connected graphs. The polytopes $H_{Del_{G_1}}(O)$ and $H_{Del_{G_2}}(O)$ are congruent if and only if G_1 and G_2 are isomorphic.*

Proof. If G_1 and G_2 are isomorphic, then we know that there exists a permutation matrix or in other words $\sigma \in \text{Aut}(S)$ such that $Q(G_2) = \sigma \cdot Q(G_1) \cdot \sigma^{-1}$. This implies that we have $b_i^{G_2} = \sigma \cdot b_{\sigma^{-1}(i)}^{G_1}$. We now claim that $H_{Del_{G_2}}(O) = \sigma \cdot H_{Del_{G_1}}(O)$. In order to see this observe that $\Delta_{\sigma_1}(G_2) = \sigma \cdot \Delta_{\sigma^{-1}\sigma_1}(G_1)$. By the definition of $H_{Del_G}(O)$, we have $H_{Del_{G_2}}(O) = \cup_{\sigma_1 \in S_{n+1}} \Delta_{\sigma_1}(G_2) = \cup_{\sigma_1 \in S_{n+1}} \sigma \Delta_{\sigma^{-1}\sigma_1}(G_1) = \sigma \cdot H_{Del_{G_1}}(O)$. By Lemma 26, we know that σ is an orthogonal transformation and hence this implies that $H_{Del_{G_1}}(O)$ and $H_{Del_{G_2}}(O)$ are congruent.

Conversely, if $H_{Del_{G_1}}(O)$ and $H_{Del_{G_2}}(O)$ are congruent then there exists an isometry $M(x) = A(x) + t$ for some orthogonal transformation A and $t \in H_0$ such that $H_{Del_{G_2}}(O) = M(H_{Del_{G_1}}(O))$. Since, A is a non-singular transformation, we know that M induces a bijection between the facets of $H_{Del_{G_1}}(O)$ and $H_{Del_{G_2}}(O)$. We also know that a facet of $H_{Del_G}(O)$ is of the form $F_{i,j}^G$ for some $0 \leq i, j \leq n$. Now, consider an arbitrary facet $F_{i,j}^{G_2}$ of $H_{Del_{G_2}}(O)$ and let the facet $M(F_{i,j}^{G_2})$ of $H_{Del_{G_1}}(O)$ be $F_{i',j'}^{G_1}$. Furthermore, we know that M induces a bijection between the vertices of $F_{i,j}^{G_2}$ and the vertices of $F_{i',j'}^{G_1}$ and that the $n-1$ -th degree d_{n-1} is conserved and by Corollary 6 this means that either $b_i^{G_2} = M(\sum_{l=0}^n b_l^{G_1} - b_{i'}^{G_1}) = M(-b_{i'}^{G_1})$ or $b_i^{G_2} = M(b_{i'}^{G_1})$. In the first case, consider the map $M'(x) = -M(x)$. Indeed M' is an orthogonal transformation and since $H_{Del_{G_1}}(O)$ is a centrally symmetric polytope, we have $H_{Del_{G_2}}(O) = M(H_{Del_{G_1}}(O))$. Hence, we may assume without loss of generality that $b_i^{G_2} = M(b_{i'}^{G_1})$. By Corollary 6, this implies that $b_j^{G_2} = M(b_{j'}^{G_1})$. Now, we know that for every edge e of facet $F_{i,j}^{G_2}$, $M(e)$ is an edge incident on $F_{i',j'}^{G_1}$. Moreover M induces a bijection between the edges of $F_{i,j}^{G_2}$ with b_i as a vertex and the edges of $F_{i',j'}^{G_1}$ with $b_{i'}$ as a vertex. We know that the edges of $F_{i,j}^G$ incident on b_i^G are $(b_i^G, b_i^G + b_k^G)$ for $k \notin \{i, j\}$. With this information, we deduce that M induces a bijection between the vertices of the form $b_k^{G_2}$ where $k \notin \{i', j'\}$ and the vertices of the form $b_k^{G_1}$ where $k \notin \{i, j\}$. Hence, M induces a permutation σ between the vertices of $H_{Del_{G_2}}(O)$ that are of the form $b_j^{G_2}$ and the vertices of $H_{Del_{G_1}}(O)$ of the form $b_j^{G_1}$ for $0 \leq j \leq n$. Hence, we have $M(\sum_{k=0}^n b_k^{G_1}) = M(O) = \sum_{k=0}^n b_k^{G_2} = O$. Hence, $t = O$ and M is an orthogonal transformation and we have $M(b_i^{G_1}) \cdot M(b_j^{G_1}) = b_{i'}^{G_2} \cdot b_{j'}^{G_2}$. Putting it together, we have a permutation $\sigma \in S_{n+1}$ such that $b_i^{G_2} \cdot b_j^{G_2} = b_{\sigma(i)}^{G_1} \cdot b_{\sigma(j)}^{G_1}$ for all integers $0 \leq i, j \leq n$. This implies that $Q(G_2)Q^t(G_2) = Q(G_2)^2 = \sigma Q^2(G_1)\sigma^{-1} = (\sigma Q(G_1)\sigma^{-1})(\sigma Q(G_1)\sigma^{-1})$. But, we know that $Q(G_1)$ and $Q(G_2)$ are positive semidefinite and hence $\sigma Q(G_1)\sigma^{-1} = \sigma Q(G_1)\sigma^t$ is also positive semidefinite. By the unique squares lemma [15], we can conclude that $\sigma Q(G_1)\sigma^{-1}$ is the unique positive semidefinite square root of $Q^2(G_2)$ and hence $Q(G_2) = \sigma Q(G_1)\sigma^{-1}$. This shows that G_1 and G_2 are isomorphic. \square

Remark 6. *There are simpler constructions that also have the property shown in Theorem 10. For example for every connected graph G on $n+1$ vertices, associate an n -dimensional simplex given by $\mathcal{S}(G) = CH(b_0, \dots, b_n)$ where b_0, \dots, b_n are the rows of the Laplacian of G . A argument similar to last part of the proof of Theorem 10 shows that: Let G_1 and G_2 be connected graphs $\mathcal{S}(G_1)$ is congruent to $\mathcal{S}(G_2)$ if and only if G_1 and G_2 are isomorphic. On the other hand, Theorem 10 is more “canonical” in the sense that the polytopes $H_{Del(G)}(O)$ have a geometric interpretation in terms of the Laplacian lattice while the simplex $\mathcal{S}(G)$ does not seem to have any direct interpretation.*

Remark 7. *We know that the lengths of the edges of $H_{Del_G}(O)$ is essentially the degrees of different vertices of G . The volume of $H_{Del_G}(O)$ is essentially the number of spanning trees of*

$H_{Del_G}(O)$. Is there such an interpretation for the volumes of the other faces of $H_{Del_G}(O)$ in the appropriate measures?

7 On the number of graphs with a given Laplacian lattice

In Lemma 16 of the previous section, we observed that the Laplacian lattice of any tree is the root lattice A_n . This observation raises the problem of counting the number of graphs that have A_n as their Laplacian lattice. The matrix-tree theorem gives an answer to the problem.

Lemma 27. *The number of graphs that have A_n as their Laplacian lattice is exactly $(n+1)^{n-1}$ i.e. the number of labelled trees on $n+1$ vertices.*

Proof. By Lemma 16 of the previous section, we know that the root lattice A_n is the Laplacian lattice of any tree on $n+1$ vertices. Conversely, any connected graph on $n+1$ vertices that is not a tree must contain at least two spanning trees and hence by Lemma 4, its Laplacian lattice must be a sublattice of A_n with covolume strictly greater than one. \square

Lemma 27 raises the following natural problem:

Given a sublattice L of A_n . Count the number of labelled connected graphs whose Laplacian lattice is L .

We denote the number of undirected connected graphs that have L as their Laplacian lattice as $N_{Gr}(L)$. Note that $N_{Gr}(L)$ is non-zero only if L is the Laplacian lattice of a connected graph. Our main result in this section is an upper bound on the number $N_{Gr}(L_G)$ for a Laplacian lattice L_G in terms of the number of different Delaunay triangulations of L_G under the simplicial distance function d_Δ . As a corollary, we show that for a Laplacian lattice L_G of a graph where every pair of vertices are connected by an edge, we have $N_{Gr}(L_G) = 1$.

From now on, when we say Delaunay triangulation, we mean that the Delaunay triangulation under the simplicial distance function d_Δ , see Subsection 2.4.3 for a discussion on Delaunay triangulation under polyhedral distance function and Subsection 3.4 for a discussion on the Delaunay triangulation of Laplacian lattices under the simplicial distance function d_Δ .

We now make precise what two triangulations are different means.

For a triangulation T , let $H_T(O)$ be the union of simplices in the triangulation that have the origin O as a vertex. We say T_1 and T_2 are the same if $H_{T_1}(O) = H_{T_2}(O)$, otherwise they are different.

Theorem 11. *Let $Del(G)$ be the Delaunay triangulation of L_G defined by the graph G under the distance function d_Δ and $N_{Del}(L_G, \Delta)$ be the number of different Delaunay triangulations of L_G , we have $N_{Gr}(L_G) \leq N_{Del}(L_G, \Delta)$.*

7.1 Proof of Theorem 11

We know that every graph provides a Delaunay triangulation of its Laplacian lattice (Theorem 6). We show that if $H_{Del_{G_1}}(O) = H_{Del_{G_2}}(O)$ then $Q(G_1) = Q(G_2)$. In other words, G can be uniquely recovered from $H_{Del_G}(O)$. This would imply that two graphs cannot give rise to the same Delaunay triangulation and Theorem 11 then follows.

Lemma 28. *Let G_1, G_2 be connected graphs, $H_{Del_{G_1}}(O) = H_{Del_{G_2}}(O)$ if and only if $Q(G_1) = Q(G_2)$.*

Proof. Indeed if $Q(G_1) = Q(G_2)$ then $H_{Del_{G_1}}(O) = H_{Del_{G_2}}(O)$. To show the converse, we describe an algorithm to uniquely recover $Q(G_1)$ from $H_{Del_{G_1}}(O)$.

Define the set C_i as follows:

$$C_i = \{p = (p_0, \dots, p_n) \in \mathbb{R}^{n+1} \mid p_i \geq 0 \text{ and } p_j \leq 0 \text{ for all } j \neq i\}.$$

Indeed, we verify that C_i is a cone. Consider the set $H_{Del_{G_1}}(O)|C_i$ of vertices of $H_{Del_{G_1}}(O)$ that are contained in the cone C_i . Pick a vertex v , say in $H_{Del_{G_1}}(O)|C_i$ that maximizes the value of the i -th coordinate.

First, a vertex with this property exists since the vertex $b_i(G_1)$ is contained in C_i since $b_{ij}(G_1)$ the j -th coordinate of $b_i(G_1)$ satisfies $b_{ij}(G_1) \leq 0$ for $i \neq j$. Furthermore, we claim that v is unique and is equal to $b_i(G_1)$. Now, assume that $v \neq b_i(G_1)$. By Lemma 17, we know that $v = b_{i_0}(G_1) + \dots + b_{i_k}(G_1)$ for some $0 \leq k \leq n-1$ and i_j s all being distinct. Denote $S_k = \{i_0, \dots, i_k\}$. Since, $b_{ij}(G_1) \leq 0$ for all $i \neq j$ we know that $v_i = b_{ii}(G_1)$ and hence, $i \in S_k$ and $b_{ij}(G_1) = 0$ for all $j \in S_k/\{i\}$. We know that $v_j \geq 0$ for all $j \in S_k$. But since $v \in C_i$, this means that $v_j = 0$ for all $j \in S_k/\{i\}$. This means that $b_{jk}(G_1) = 0$ for all $j \in S_k/\{i\}$ and $k \in (V(G)/S_k) \cup \{i\}$. Hence, there are at least two components of G that are not connected, namely the induced subgraphs of vertices $S_k/\{i\}$ and $(V(G)/S_k) \cup \{i\}$. This contradicts our assumption that G is connected. This shows that if $H_{Del_{G_1}}(O) = H_{Del_{G_2}}(O)$ then $b_i(G_1) = b_i(G_2)$ for all $0 \leq i \leq n$ and hence $Q(G_1) = Q(G_2)$. This concludes the proof of the lemma. □

This shows that each graph G with $L_G = L$ contributes to a different Delaunay triangulation of L and hence, $N_{Gr}(L_G) \leq N_{Del}(L_G, \Delta)$. This concludes the proof of Theorem 11.

We know from the results in [2] that multigraphs where every pair of vertices are connected by an edge have a unique Delaunay triangulation. As a corollary we obtain:

Corollary 7. *If L_G is the Laplacian lattice of multigraph G such that every pair of vertices are connected by an edge, then $N_{Gr}(L_G) = 1$.*

8 Covering and Packing problems on the Laplacian lattice

Covering and Packing problems on lattices have been widely studied, see Conway and Sloane [8] for a general introduction and Zong and Talbot [30] or Martinet [20] for a more specialised treatment of the subject.

Given a lattice L and a convex body \mathcal{P} , we study to how “optimally” do L -translates of \mathcal{P} pack or cover \mathbb{R}^{n+1} . For a sublattice L of A_n , we define the packing density $\gamma_C(L)$ and the covering density $\theta_C(L)$ as:

$$\begin{aligned} \gamma_C(L) &= \text{Pac}_C(L)/((n+1)C\text{vol})^{1/n}(L) \\ \theta_C(L) &= \text{Cov}_C(L)/((n+1)C\text{vol})^{1/n}(L) \end{aligned} \tag{9}$$

where $\text{Pac}_C(L)$ and $\text{Cov}_C(L)$ is the packing and covering radius of L with respect to C and $C\text{vol}(L)$ is the covolume of the lattice with respect to A_n .

Remark that in the standard definition of packing and covering density, the volume of the lattice L appears in the place of $n+1$ times the covolume of L . Observe that the two notions are “equivalent” up to a factor that depends only on the dimension of L and are interchangeable since we are interested in determining lattices with good packing and covering densities in a

given dimension and the order of the quotient group A_n/L is equal to the ratio of the volumes of A_n and L i.e. $Vol(L) = Vol(A_n) \cdot |A_n/L|$ (see Lecture V, Theorem 20 of Siegel [27] for a proof).

The lattice packing and covering problem respectively is to find lattices that pack \mathbb{R}^n most densely and cover \mathbb{R}^n most economically i.e. lattices that maximise $\gamma_C(\cdot)$ and minimise $\theta_C(\cdot)$. Explicit constructions of lattices that solve the sphere packing and covering problems are known only for lower dimensions. In two dimensions, the hexagonal lattice produces the densest sphere packing.

We will use the fact that the covering and packing radius of the Laplacian lattice have a combinatorial interpretation in terms of the underlying graph to obtain a formula for the covering and packing density of the Laplacian lattice. We will use this information to show that in the space of Laplacian lattices of undirected connected graphs, the Laplacian lattices of graphs that are highly connected such as Ramanujan graphs have relatively good packing and covering properties.

As a direct consequence of the formulas for the packing and covering radius (Theorem 7 and Theorem 8) we obtain:

Corollary 8. *The packing radius and covering radius of the Laplacian lattice L_G respectively are:*

$$\gamma_\Delta(L_G) = MC_1(G) / ((n+1) \left(\prod_{i=1}^n \lambda_i \right)^{1/n}) \quad (10)$$

$$\theta_\Delta(L_G) = \left(\sum_{i=1}^n \lambda_i \right) / (2(n+1) \left(\prod_{i=1}^n \lambda_i \right)^{1/n}) \quad (11)$$

where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ are the non-zero eigenvalues of the Laplacian matrix of G .

This immediately gives a lower bound for the covering density of a Laplacian lattice:

Theorem 12. (Lower bounds for the covering density of Laplacian lattices) *The covering density of the Laplacian lattice is at least $n/2(n+1)$.*

Proof. By Corollary 8, we know that $\theta_\Delta(L_G) = (\sum_{i=1}^n \lambda_i) / (2(n+1) (\prod_{i=1}^n \lambda_i)^{1/n})$ where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ are the non-zero eigenvalues of the Laplacian matrix. Now we use the fact that the Laplacian matrix is a positive semidefinite matrix along with the AM-GM inequality to obtain that $\theta_\Delta(L_G) \geq n/2(n+1)$. \square

We consider the problem of minimising the covering density and maximising the packing density of the Laplacian lattice over all connected graphs with a given number of vertices. First we consider the covering density case. Suppose that $\lambda_1, \dots, \lambda_n$ are arbitrary positive real numbers then the quantity $\text{Cov}_\Delta(L_G)$ is minimised if $\lambda_1 = \lambda_2 = \dots = \lambda_n$. But then in our case, these numbers are eigenvalues of the Laplacian matrix. Nevertheless, we would like the eigenvalues of the Laplacian matrix to be “clustered”. This suggests that graphs with good expansion properties would be suitable. To make this intuition precise, we need the notion of a Ramanujan graph, see the survey of Horory et al. [16] for a more detailed discussion on the topic.

Definition 14. *A d -regular graph is called a Ramanujan graph if $\lambda^A(G) \leq 2\sqrt{d-1}$, where $\lambda^A(G) = \max\{|\lambda_2^A|, |\lambda_{n+1}^A|\}$ and $d = \lambda_1^A \geq \dots \geq \lambda_{n+1}^A$ are the eigenvalues of the adjacency matrix of G .*

Using the fact that $\lambda_i = d - \lambda_i^A$ we have:

Lemma 29. *The non-zero eigenvalues of the Laplacian matrix of a Ramanujan graph are located in the interval $[d - 2\sqrt{d-1}, d + 2\sqrt{d-1}]$.*

Suppose that the graph is a Ramanujan graph, then we know that the eigenvalues of its Laplacian matrix are concentrated around d (the degree of the graph) in an interval of width $2\sqrt{d-1}$. More precisely, we have: $d - 2\sqrt{d-1} \leq \lambda_i \leq d + 2\sqrt{d-1}$ for every non-zero eigenvalue of the Laplacian. Using this information, we will obtain an upper bound on the covering density of a Ramanujan graph.

Lemma 30. *Let G be a d -regular Ramanujan graph, we have $d + 2\sqrt{d-1} \geq (\prod_{i=1}^n \lambda_i)^{1/n} \geq d - 2\sqrt{d-1}$*

Proof. Since G is a d -regular Ramanujan graph we have $d - 2\sqrt{d-1} \leq \lambda_i \leq d + 2\sqrt{d-1}$ for every i from $1 \dots n$. Using this information, we have $(d + 2\sqrt{d-1}) \geq (\prod_{i=1}^n \lambda_i)^{1/n} \geq (d - 2\sqrt{d-1})$. \square

Theorem 13. (Covering Density of Ramanujan graphs) *Let G be a d -regular Ramanujan graph then $\theta_\Delta(L_G) \leq (\frac{d}{4(d-2\sqrt{d-1})})$.*

Proof. We have $\sum_{i=1}^n \lambda_i = (n+1) \cdot d/2$ and by Lemma 30 we have $\prod_{i=1}^n \lambda_i \geq d - 2\sqrt{d-1}$. By Corollary 8, we know that $\theta_\Delta(L_G) = (\sum_{i=1}^n \lambda_i) / (2(n+1)(\prod_{i=1}^n \lambda_i)^{1/n})$ where $\lambda_1 \leq \lambda_2 \dots \leq \lambda_n$ are the non-zero eigenvalues of the Laplacian matrix. Hence, we obtain $\theta_\Delta(L_G) \leq (\frac{d}{4(d-2\sqrt{d-1})})$. \square

Remark 8. *Note that we crucially use the fact that the eigenvalues of a Ramanujan graph are concentrated in a small interval. It is not clear if we can obtain upper bounds on the covering density for a general graph.*

We do not know if the converse of Theorem 13 also holds. More precisely,

Question 1. *Suppose that the covering density of the Laplacian lattice of a graph is upper bounded by a suitably chosen constant c , then is it true that the graph is Ramanujan?*

We now consider the problem of maximising the packing density of the Laplacian lattice. The formula for packing density presented in Theorem 8 suggests that graphs that maximise the packing density have high minimum cut and a relatively small number of spanning trees. We will now provide a lower bound on the packing density of Ramanujan graphs.

Lemma 31. *Let G be a d -regular Ramanujan graph, then $\gamma_\Delta(G) \geq \frac{(d-2\sqrt{d-1})}{2(n+1)(d+2\sqrt{d-1})}$.*

Proof. By Corollary 8 we have: $\gamma_\Delta(G) = \frac{\text{MC}_1(G)}{(n+1)(\prod_{i=1}^n \lambda_i)^{1/n}}$. By Lemma 30 we have: $\prod_{i=1}^n \lambda_i \leq d + 2\sqrt{d-1}$. We now obtain a lower bound on $\text{MC}_1(G)$. We observe that the size of the cut S can be written as $u_S \cdot Q(G) \cdot u_S^t / 2$ where u_S is the indicator vector of S i.e.

$$u_{Si} = \begin{cases} 1, & \text{if the vertex with index } i \text{ is in } S, \\ -1, & \text{otherwise.} \end{cases}$$

We now observe that:

$$\min_{S \subseteq \{V, \emptyset\}} \frac{u_S \cdot Q(G) \cdot u_S^t}{2} \geq \min_{u \in \mathbb{S}^1} \frac{u \cdot Q(G) \cdot u^t}{2} \cdot \min_{S \subseteq \{V, \emptyset\}} u_S \cdot u_S \geq \min_{u \in \mathbb{S}^1} \frac{u \cdot Q(G) \cdot u^t}{2} = \lambda_n / 2$$

Now since the graph is a Ramanujan graph we know that $\lambda_n \geq d - 2\sqrt{d-1}$. Hence, $\gamma_\Delta(G) \geq \frac{d-2\sqrt{d-1}}{2(n+1)(d+2\sqrt{d-1})}$. \square

A natural question is that whether the lower bound on the packing density that we obtained for the Laplacian lattice of a Ramanujan graph is the best possible. Note that a trivial upper bound for $\gamma_\Delta(G)$ is d for a d -regular graph G . It would be interesting to study the converse of this question, namely, suppose that the Laplacian lattice of a d -regular graph has a high packing density then does the graph have “high” connectivity?

9 Appendix

9.1 Sequences of Lattices and their limit

A sequence of lattices $\{L_n\}$ is said to **converge** to a lattice L_ℓ if for every $\delta > 0$ and $q \in L_\ell$ there exists a positive integer $N(\delta, q)$ and a family of bijective maps $\phi_N : L_\ell \rightarrow L_N$ for $N \geq N(\delta, q)$ such that $\|q - \phi_N(q)\|_2 < \delta$ for all $N \geq N(\delta, q)$. Similarly, a sequence of lattices $\{L_n\}$ is said to **uniformly converge** to a limit L_ℓ if for every $\delta > 0$ there exists an integer $N(\delta)$ and a family of bijective maps $\phi_N : L_\ell \rightarrow L_N$ for $N \geq N(\delta)$ such that $\|q - \phi_N(q)\|_2 < \delta$ for all $q \in L_\ell$ and for all $N \geq N(\delta)$. Note that since convex polytopes are topologically equivalent to Euclidean balls, the notion of convergence of a sequence of lattices with respect to polyhedral distance functions is equivalent to the above notion.

Typically, we take a basis \mathcal{B} of the lattice and add an infinitesimal perturbation \mathcal{B}^ϵ to it and consider the lattice generated by the perturbed lattice. As ϵ tends to zero, the sequence of lattices converges to the lattice generated by \mathcal{B} , but not necessarily uniformly. Observe that there is a natural bijection ϕ_ϵ from L_ℓ to L_ϵ induced by the basis \mathcal{B} defined as $\phi_\epsilon(\mathcal{B}\alpha) = \mathcal{B}^\epsilon\alpha$. In the following lemma we prove that the shortest vector and packing radius are preserved under the limit, see the book of Gruber and Lekkerkerker [13] for a more detailed treatment of lattices under perturbation.

Lemma 32. *Consider a sequence of lattices $\{L_n\}$ that converge to a lattice L_ℓ . For any convex polytope \mathcal{P} , the length of the shortest vector and packing radius under the distance function $d_{\mathcal{P}}$ is preserved under limit. More precisely we have:*

1. $\lim_{n \rightarrow \infty} \nu_{\mathcal{P}}(L_n) = \nu_{\mathcal{P}}(L_\ell)$
2. $\lim_{n \rightarrow \infty} \text{Pac}_{\mathcal{P}}(L_n) = \text{Pac}_{\mathcal{P}}(L_\ell)$.

Proof. i. Given an $\epsilon > 0$, we show that there exists a positive integer $n(\epsilon)$ such that $|\nu_{\mathcal{P}}(L_\ell) - \nu_{\mathcal{P}}(L_n)| \leq \epsilon$ for all $n \geq n(\epsilon)$. Consider the set of shortest vectors M_1 of the lattice L_ℓ . Let d be the difference between the norm of the shortest vector and second shortest vector in the distance function $d_{\mathcal{P}}$. Let δ be the minimum of $d/2$ and $\epsilon/2$. Take a ball of radius R much larger than $\nu_{\mathcal{P}}(L_\ell)$ centered at the origin, in fact taking R to be $2\nu_{\mathcal{P}}(L_\ell) + d$ suffices. Using our assumption that $\{L_n\}$ converges to L_ℓ and the fact that number of points in L_ℓ that are contained inside the ball of radius R centered at the origin is finite, we know that there exists a positive integer $n(\delta)$ and a bijection $\phi_n : L_\ell \rightarrow L_n$ such that for $n \geq n(\delta)$ we have $d_{\mathcal{P}}(q, \phi_n(q)) < \delta$ for points q in L_ℓ that are contained in the ball of radius R . For a point p , we call $d_{\mathcal{P}}(O, p)$ the norm of the point. Now we claim that for all $n \geq n(\delta)$ any shortest vector of L_n belongs to $\mathcal{P}(q', \delta)$ where $q' \in M_1$. The argument is as follows: any element in L_n for $n \geq n(\delta)$ that does not belong to $\mathcal{P}(q', \delta)$ for $q' \in M_1$ must have norm strictly greater than $\nu_{\mathcal{P}}(L_\ell) + d/2$. On the other hand, we know that there exists an element of L_n that is

contained in $\mathcal{P}(q', \delta)$ for some $q' \in M_1$ and hence has norm at most $\nu_{\mathcal{P}}(L_\ell) + \delta$. Since $\delta \leq d/2$, we arrive at a contradiction. We finally note that every point contained in the ball $\mathcal{P}(q', \delta)$ for $q' \in M_1$ has norm between $\nu_{\mathcal{P}}(L_\ell) + \delta$ and $\nu_{\mathcal{P}}(L_\ell) - \delta$. Hence, $|\nu_{\mathcal{P}}(L_n) - \nu_{\mathcal{P}}(L_\ell)| \leq 2\delta \leq \epsilon$ for all $n \geq n(\delta)$.

ii. For the packing radius, the argument is essentially the same as the argument for shortest vectors except that in this case we carry out the argument on the \mathcal{P} -midpoints of the lattice points with the origin rather than on the lattice points. We consider the \mathcal{P} -midpoints of every point of L_ℓ with the origin. Let M_1 be the set of \mathcal{P} -midpoints of every point in L_ℓ that define its packing radius. Let d be the difference between the packing radius and the norm of \mathcal{P} -midpoints that are closer to the origin expect to the \mathcal{P} -midpoints that define the packing radius. Take δ to be the minimum of $d/2$ and $\epsilon/2$. Take a ball of radius R much larger than $\text{Pac}_{\mathcal{P}}(L_\ell)$ centered at the origin. Since $\{L_n\}$ converges to L_ℓ and since the \mathcal{P} -midpoints vary continuously with perturbation, we know that there exists a integer $n(\delta)$ and a bijection $\phi_n : L_\ell \rightarrow L_n$ such that $d_{\mathcal{P}}(b, \phi_n(b)) < \delta$ for \mathcal{P} -midpoints in the ball of radius R . Now we claim that for $n \geq n(\delta)$ the \mathcal{P} -midpoint that defines the packing radius of L_n must be contained in $\mathcal{P}(b, \delta)$ for some $b \in M_1$. This follows from the fact that for $n \geq n(\delta)$ any element in L_n that does not belong to $\mathcal{P}(b', \delta)$ where $b' \in M_1$ must have norm strictly greater than $\text{Pac}_{\mathcal{P}}(L_\ell) + d/2$. On the other hand, we know that the bisectors of elements of L_n that are contained in $\mathcal{P}(q', \delta)$ have norm at most $\text{Pac}_{\mathcal{P}}(L_\ell) + \delta$. Since $\delta \leq d/2$, we arrive at a contradiction. We finally note that every element of $\mathcal{P}(q', \delta)$ for $q' \in M_1$ contained in the ball has norm between $\text{Pac}_{\mathcal{P}}(L_\ell) + \delta$ and $\text{Pac}_{\mathcal{P}}(L_\ell) - \delta$. Hence, $|\text{Pac}_{\mathcal{P}}(L_n) - \text{Pac}_{\mathcal{P}}(L_\ell)| \leq 2\delta \leq \epsilon$ for all $n \geq n(\delta)$. □

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